2 Filters and Completeness

Definition 2.1 *Let S be a set. A* **filter on** *S is a family F of subsets of S which is inclusion-stable, stable under finite intersections and nontrivial, i.e.*

- *If $f \in F$ and $f \subset g$, then $g \in F$,*
- *If $f, g \in F$, then $f \cap g \in F$,*
- *$\emptyset \neq F$.*

If $F_1, F_2 \subset P(M)$ are two filters with $F_1 \subset F_2$, we say that F_2 **is finer than** *F_1.*

The principal example of a filter is given by the family N_p of neighborhoods of a fixed point of a topological space. A further, not so important, example is the filter F_A of all subsets of S containing the fixed subset $A \neq \emptyset$. The last example is important to understand filters as generalizations of sequences: For every sequence s in a set S, we define the filter F_s as the family of all subsets of S containing the image of all but finitely many terms of s.

Definition 2.2 (Convergence of filters) *Let F be a filter in a topological space T. We say that F converges to a point $p \in T$ iff F is finer than N_p. In this case, p is called a* **limit** *of F.*

Exercise: Show that a sequence a converges to a point p if and only if F_a converges to A!
Exercise: Show that a filter on a Hausdorff topological space cannot have more than one limit!
Exercise: Show that N_p converges to p!

Definition 2.3 *Let A be a subset of a tvs V. A filter F on A is called* **Cauchy** *iff for every $U \in N_0$ there is a $f \in F$ with $f - f \in U$.*

Exercise: Show that for a Cauchy sequence s, the filter F_s is Cauchy, too.

Theorem 2.4 *Let V be a tvs. Let F be a Cauchy filter on V and G be a filter on V finer than F. Then G is Cauchy, too.*

Proof: trivial. □

Theorem 2.5 *Every convergent filter is Cauchy.*

Proof. First we show that for any p, the filter N_p is Cauchy. Thus if U is a neighborhood of 0, then we can find $V \in N_0$ with $U \supset V - V \subset U = V + p - (V + p)$. Then the statement follows from the preceding theorem. □

Definition 2.6 *A tvs is called* **complete** *if every vector-Cauchy filter on V converges to a point in V. It is called* **sequentially complete** *if every vector-Cauchy sequence on V converges to a point in V.*

In the light of the first exercise we see that sequential completeness follows from completeness. The inverse is wrong in general.

Theorem 2.7 *In a Hausdorff tvs, every complete subset is closed.*

Proof: Exercise. □

Theorem 2.8 *In a complete tvs, every closed subset is complete.*

Proof: Exercise. $\square$

Definition 2.9 *Let A be a vector subspace of a tvs V. The* **sequential closure** $\overline{A}^s$ **of** A **in** V *consists of all points $p \in V$ such that there is a sequence $a_n \in A$ with* $\lim_{n\to\infty} a_n p$.

Theorem 2.10 *Let A be a vector subspace of a tvs V. The topological closure $\overline{A}$ coincides with the set of points $p \in V$ such that there is a filter in A wih limit p.*

Now, we adopt the definition of completion from the Treves book!

Definition 2.11 *Let A be a tvs. The* **sequential completion** $\overline{A}^s$ **of** A *is defined as the sequential closure of A in $\overline{A}$.*

Theorem 2.12 *The sequential completion on every tvs V is sequentially complete, and V is sequentially densely embedded in $\overline{V}^s$.*